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Continuity of Metric Projections onto Subspaces and Openness of Quotient Maps on Unit Balls

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It is easy to find a three-dimensional space such that the set-valued metric projection onto a linear subspace fails to be lower semicontinuous. There are many papers on the lower semicontinuity of set-valued metric projections (e.g., [4, 5, 13]). Conditions which guarantee the Lipschitz continuity of single-valued metric projections are formulated, for example, in [2, 12]. The relative openness of affine maps on convex sets has been treated as well (see, e.g., [3, 7, 15, 17, 18]), but so far no one has pointed out its relation to the problems mentioned above.

In this paper we prove that for any proximinal linear subspace M of a normed linear space X the metric projection of X onto M is lower semicontinuous if and only if the quotient map $X \to X/M$ is relatively open on the closed unit ball of X. A similar assertion concerning the pointwise Lipschitz continuity of metric projections is stated as well. The results are used to discuss continuity of some metric projections in both sequence and function spaces.

If not stated otherwise our notation and terminology are those of [8]. By c_0, l_1, and l_∞ we mean the sequence spaces $c_0(\Gamma)$, $l_1(\Gamma)$, and $l_\infty(\Gamma)$, respectively, with Γ consisting of the positive integers. Let X be a real normed linear space, M be a closed linear subspace, and A a subset of X. A mapping Q from X into another normed linear space is said to be relatively open on A at $x \in A$ if for any neighbourhood V of x in A, $Q(V)$ is a neigbourhood of $Q(x)$ in $Q(A)$. Further, Q is said to be relatively open on A if it is relatively open on A at any $x \in A$. For any $\varepsilon > 0$ and $x \in A$, the set $\{a \in A: \|a - x\| < \varepsilon\}$ will be called ε-neighbourhood of x in A. The set-valued metric projection P_M of X onto M is defined by

$$P_M(x) = \{p \in M: \|x - p\| \leqslant \|x - m\| \text{ for any } m \in M\}, \qquad x \in X.$$

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The set $\{x \in X: 0 \in P_M(x)\}$ will be denoted by $P_M^{-1}(0)$. The set M is said to be proximinal if $P_M(x) \neq \emptyset$ for any $x \in X$; M is said to be Čebyšev if P_M is a single-valued mapping defined on X. The projection P_M is said to be lower semicontinuous (lsc) at $x_0 \in X$ if for any open set $G \subset X$ with $P_M(x_0) \cap G \neq \emptyset$ the set $\{x \in X: P_M(x) \cap G \neq \emptyset\}$ is a neighbourhood of x_0 in X. The projection P_M is said to be lsc if it is lsc at any point of X.

We start with a definition and a plain remark.

(1) Definition. We shall say that P_M is pointwise Lipschitz lsc at x_0 if there is a constant K such that for any $x \in X$ and any $m_0 \in P_M(x_0)$ there exists $m \in P_M(x)$ such that $\|m - m_0\| \leqslant K\|x - x_0\|$. We shall say that P_M is pointwise Lipschitz lsc if it is such at any $x_0 \in X$.

(2) Remark. Let M be a proximinal linear subspace of a normed linear space X. To show that P_M is lsc or pointwise Lipschitz lsc, it suffices to verify that P_M has that property on the norm one elements of $P_M^{-1}(0)$. Let us take $x_0 \in M$ at first. Then $P_M(x_0) = \{x_0\}$ and for any $m \in P_M(x)$, $x \in X$, we have

$$\|m - x_0\| \leqslant \|m - x\| + \|x - x_0\| \leqslant 2\|x_0 - x\|.$$

Particularly, P_M is pointwise Lipschitz lsc at x_0. Further, let $x_0 \in X$ and $m_0 \in P_M(x_0)$ be arbitrary. Then $x_0 - m_0$ is an element of $P_M^{-1}(0)$ and $P_M(x - m_0) = \{m - m_0: m \in P_M(x)\}$ for any $x \in X$. Finally, P_M is positively homogeneous. The main result is

(3) Theorem. *Let X be a normed linear space, M a proximinal linear subspace of X, $Q: X \to X/M$ the quotient map, U the closed unit ball of X, and $x_0 \in P_M^{-1}(0)$ a point of norm one. We define $V = \{x_0 - m: m \in P_M(x_0)\}$. Then P_M is lsc at x_0 if and only if Q is relatively open on U at any point of V. Moreover, the following two conditions are equivalent*:

(i) *P_M is pointwise Lipschitz lsc at x_0 (see Definition (1))*;

(ii) *there exists $c > 0$ such that for any $x \in V$ and any $\varepsilon \in (0, 2]$, Q maps the ε-neighbourhood of x in U at least onto the $c\varepsilon$-neighbourhood of $Q(x)$ in $Q(U)$.*

(4) Corollary. *Let X, M, Q, and U be as in (3). Then P_M is lsc if and only if Q is relatively open on U. Suppose in addition that M is Čebyšev. Then P_M is pointwise Lipschitz continuous if and only if for any $x \in U$ there is $c > 0$ such that for any $\varepsilon \in (0, 2]$, Q maps the ε-neighbourhood of x in U at least onto the $c\varepsilon$-neighbourhood of $Q(x)$ in $Q(U)$.*

Proof of (3). Let $m_0 \in P_M(x_0)$ and $0 < \delta \leqslant \varepsilon \leqslant 2$ be such that if $x \in X$ and $\|x - x_0\| < \delta$, then there exists $m \in P_M(x)$ with $\|m - m_0\| < \varepsilon$. We show that Q maps the 2ε-neighbourhood of $x_0 - m_0$ in U at least onto the δ-neighbourhood of $Q(x_0)$ in $Q(U)$.

Take an arbitrary $y \in Q(U)$ with $\|y - Q(x_0)\| < \delta$. As Q is quotient map, there exists $x \in Q^{-1}(y)$, $\|x - x_0\| < \delta$. As assumed, we can choose $m \in P_M(x)$, $\|m - m_0\| < \varepsilon$. Since m is best approximation of x in M, $\|x - m\| = \|Q(x)\| \leqslant 1$, hence $x - m$ is an inverse image of y in U. Finally, $\|(x-m) - (x_0 - m_0)\| \leqslant \|x - x_0\| + \|m - m_0\| < 2\varepsilon$.

Thus we have proved that if P_M is lsc at x_0, then Q is relatively open on U at any point of V. Moreover, the implication (i) $\Rightarrow$ (ii) follows from this since we can put $c = (2K)^{-1}$, where $K \geqslant 1$ is a constant as in (1).

To prove the inverse implication let $m_0 \in P_M(x_0)$, $\varepsilon \in (0, 2]$, and $\delta \in (0, 1/2\varepsilon]$ be such that Q maps the ε-neighbourhood of $x_0 - m_0$ in U at least onto the 2δ-neighbourhood of $Q(x_0)$ in $Q(U)$. Let $x \in X$ be such that $\|x - x_0\| < \delta$. We will find $m \in P_M(x)$ with $\|m - m_0\| < 2\varepsilon$. We put $Q(x_0) = y_0$, $Q(x) = y$, and $\|y\| = \alpha$. There is $0 \in P_M(x_0)$, hence $\|y_0\| = 1$. Of course, $\|y - y_0\| < \delta$, whence $|\alpha - 1| < \delta$ (and particulary $\alpha \neq 0$ since $\delta \leqslant 1$). Define $y_1 = \alpha^{-1} y$. Then $\|y_1 - y_0\| \leqslant \|y_1 - y\| + \|y - y_0\| < 2\delta$, hence we can choose $x_1 \in Q^{-1}(y_1) \cap U$ such that $\|x_1 - (x_0 - m_0)\| < \varepsilon$. Setting $m = x - \alpha x_1$ we claim that m is a desired element.

First $Q(m) = y - \alpha y_1 = 0$, whence $m \in M$. Further, $m \in P_M(x)$ for $\|x - m\| = \alpha \|x_1\| \leqslant \alpha = \|Q(x)\|$. Finally, we can write $m - m_0 = (x - x_0) + (x_0 - x_1 - m_0) + (x_1 - \alpha x_1)$, hence $\|m - m_0\| < \delta + \varepsilon + \delta \leqslant 2\varepsilon$.

Thus we have proved that if Q is relatively open on U at any point of V then P_M is lsc at x_0.

Suppose further that the condition (ii) is fulfilled with a certain constant $c \leqslant 1$. Let $x \in X$ be such that $0 < \|x - x_0\| \leqslant 2/3c$. Put $\varepsilon = 3c^{-1}\|x - x_0\|$ and $\delta = 1/2c\varepsilon$. Then $\|x - x_0\| < \delta$ and applying the fact just proved we obtain $\operatorname{dist}(P_M(x), m_0) < 6c^{-1}\|x - x_0\|$. Now let $x \in X$ be such that $\|x - x_0\| > 2/3c$ and $m \in P_M(x)$ be arbitrary. Since $\|m - x\| = \|Q(x)\| \leqslant \|x\| \leqslant 1 + \|x - x_0\|$ and $\|m_0 - x_0\| \leqslant \|x_0\| = 1$, we get $\|m - m_0\| \leqslant 2 + 2\|x - x_0\| < (3c^{-1} + 2)\|x - x_0\|$ which completes the proof of the implication (ii) $\Rightarrow$ (i).

Proof of (4). We show that Q is relatively open on U at any $x \in U$ such that either $\|x\| < 1$ or $0 \notin P_M(x)$. If x is such a point, then the norm of $y = Q(x)$ is less than one. Choose $u \in Q^{-1}(y)$, $\|u\| < 1 - 3c$ with some $c > 0$. For any $\varepsilon \in (0, 2]$ define $v = v(\varepsilon) = x + 1/3\varepsilon(u - x)$. Then $\|v - x\| \leqslant 2/3\varepsilon$ and $\|v\| \leqslant (1 - 1/3\varepsilon)\|x\| + 1/3\varepsilon\|u\| < 1 - c\varepsilon$. Hence the ε-neighbourhood of x in U contains the $c\varepsilon$-neighbourhood of v in X which is carried by Q onto the $c\varepsilon$-neighbourhood of y in X/M. The rest of the proof follows from Theorem (3) and Remark (2).

It is not difficult to show that if U is a finite-dimensional polyhedron then any affine map on U is relatively open on U. To generalize this fact we introduce

(5) DEFINITION. Let U be the closed unit ball of a normed linear space X and $u \in U$ be an arbitrary point of norm one.

(i) We shall say that U is polyhedral ((PH) in short) at u if there exists $\delta > 0$ and a finite set $F \subset X^*$, $\|f\| = f(u) = 1$ for $f \in F$, such that whenever $x \in X$, $\|x - u\| < \delta$ and $f(x) \leqslant 1$ for all $f \in F$, then $x \in U$. We shall say that X is a (PH)-space if U is (PH) at any point of norm one.

(ii) We shall say that U is quasi-polyhedral ((QP) in short) at u with δ ($\delta > 0$) if any point of the δ-neighbourhood of u in U belongs to a line segment $[u, x]$ of length at least δ with some $x \in U$. When it is not necessary to point out the value of δ, we shall say just that U is (QP) at u. It can be easily seen that X is a (QP)-space in the sense of [1] if and only if U is (QP) at any $u \in U$ of norm one.

Clearly, if the closed unit ball U of a normed linear space X is (PH) at a point u, then it is (QP) at u (with the same constant δ as in (5) (i)), whence u is not a cluster point of extreme points of U. Consequently, a finite-dimensional space X is a (PH)-space if and only if U is a polyhedron (i.e., the set of extreme points of U is finite).

For any set $\Gamma \neq \varnothing$, the sequence space $c_0(\Gamma)$ is a (PH)-space (notice that for any $u \in c_0(\Gamma)$ with $\|u\| = 1$ one can take $\delta = 1 - \sup\{|u_\gamma| : \gamma \in \Gamma, |u_\gamma| < 1\}$). The product of a family $\{X_\gamma\}_{\gamma \in \Gamma}$ of (PH)-spaces in the sense of $c_0(\Gamma)$ is a (PH)-space. Clearly, any linear subspace of a (PH)-space is again a (PH)-space.

The space c_0 with the modified norm $\|(x_n)\| = |x_1| + \sup\{|x_n| : n > 1\}$ is an example of a (QP)-space which is not a (PH)-space.

We proceed with a fact that can be seen forthwith.

(6) *Remark.* Let M be a closed linear subspace of a normed linear space X, U the closed unit ball of X, and Q: $X \to X/M$ the quotient map. Then M is proximinal if and only if $Q(U)$ is closed (i.e., for any $y \in X/M$ of norm one there is at least one element $x \in Q^{-1}(y)$ of the same norm). Further, M is Čebyšev if and only if for any $y \in X/M$ of norm one there is exactly one inverse image of the same norm.

The idea of the fist part of the following lemma comes from [17].

(7) LEMMA. *Let X, M, Q, and U be as in Theorem* (3), $W = Q(U)$, $y \in W$ *satisfy* $\|y\| = 1$, *and* $V = U \cap Q^{-1}(y)$.

(i) *If W is (QP) at y with some $\delta > 0$, then for any $\varepsilon \in (0, 2]$ and $x \in V$, Q maps the ε-neighbourhood of x in U at least onto the $1/2\delta\varepsilon$-neighbourhood of y in W.*

(ii) *If there exists $x \in V$ such that U is (QP) at x and Q is relatively open on U at x, then W is (QP) at y.*

Proof. Notice that W is closed by (6). To prove (i) let v be an arbitrary point of W; we shall find a $u \in U \cap Q^{-1}(v)$ within the distance $2\delta^{-1}\|v-y\|$ from x. Since W is (QP) at y with δ, there is $w \in W$ with $\|w-y\| \geq \delta$ such that $v = y + t(w-y)$ with $t \in [0, 1]$. Note that $t \leq \delta^{-1}\|v-y\|$. Now choose an arbitrary inverse image $\bar{x}$ of w in U and put $u = x + t(\bar{x} - x)$.

To prove (ii) suppose that U is (QP) at some $x \in V$ with a $\delta > 0$ and that Q maps the $1/2\delta$-neighbourhood of x in U at least onto the ε-neighbourhood of y in W for some $\varepsilon > 0$. Then any element $v_1 \in W$ with $0 < \|v_1 - y\| < \varepsilon$ is the midpoint of a nontrivial linear segment $[y, v_2]$ for some $v_2 \in W$. If $\|v_2 - y\| < \varepsilon$, we can proceed analogously until $\|v_n - y\| \geq \varepsilon$ with $v_n \in W$, where $v_i = 1/2(y + v_{i+1})$ for $i = 1, 2,..., n-1$.

The following elementary lemma will be used later.

(8) LEMMA. *Let X, Y be normed linear spaces, $Q: X \to Y$ a linear open mapping, $F \subset X^*$ a finite set, and H the intersection of halfspaces $\{x \in X: f(x) \leq 0\}$ for $f \in F$. Then Q is relatively open on H at 0.*

Proof. Observe at first that Q is relatively open on $H_f = f^{-1}(0)$ for any $f \in F$. To see this, denote $Z = Q(H_f)$ and consider the case $Z \neq Y$ at first. Then $Q^{-1}(z) \subset H_f$ for any $z \in Z$ and the assertion is trivial. Now, suppose that $Z = Y$. Let $x_1 \in f^{-1}(1)$ be arbitrary and $x_0 \in H_f$ be such that $Q(x_0) = Q(x_1)$. Since for any $x \in X$ the point $x + f(x)(x_0 - x_1)$ is an inverse image of $Q(x)$ in H_f of norm $(1 + \|f\| \|x_0 - x_1\|) \|x\|$ at most, it is enough to use the openness of Q on X.

Further, let $y \in Q(H)$ be close enough to 0. There exists $x \in Q^{-1}(y)$ close to 0 in X. Suppose that $x \notin H$. We choose some $\bar{x} \in Q^{-1}(y) \cap H$ and define

$$s = \sup\{t: \bar{x} + t(x - \bar{x}) \in H\} \qquad \text{and} \qquad x_0 = \bar{x} + s(x - \bar{x}).$$

Then $Q(x_0) = y$ and $x_0 \in H \cap H_f$ for some $f \in F$. We proceed by induction on the cardinality of F. If $F = \{f\}$, then $H_f \subset H$ and we can use the relative openness of Q on H_f to find an inverse image of $y = Q(x_0) \in Q(H_f)$ close to 0 in H. Let card $F > 1$. Considering the restrictions of Q on H_f ($f \in F$) we can suppose the induction hypothesis: for any $f \in F$, Q is relatively open on $H_f \cap H$ at 0. Then the relative openness of Q on H at 0 follows immediately from the arguments mentioned above and from the induction hypothesis since we have proved in fact that for any $y \in Q(H)$ either $Q^{-1}(y) \subset H$ or $y \in Q(H \cap H_f)$ for some $f \in F$.

(9) THEOREM. *Let M be a proximinal linear subspace of a normed linear space X and $x_0 \in P_M^{-1}(0)$ a point of norm one such that the closed unit ball of X is (PH) at x_0. Then P_M is pointwise Lipschitz lsc at x_0.*

(10) COROLLARY. *Let X be a linear subspace of c_0 and M be a proximinal linear subspace of X. Then the metric projection of X onto M is pointwise Lipschitz lsc.*

Proof of (9). Let U and Q be as in Theorem (3). Since U is (PH) at x_0, there exists a $\delta > 0$ and a finite set $F \subset X^*$, $\|f\| = f(x_0) = 1$ for $f \in F$, such that the δ-neighbourhood of x_0 in the set $E = \{x \in X: f(x) \leqslant 1 \text{ for all } f \in F\}$ coincides with the δ-neighbourhood of x_0 in U. Applying Lemma (8) with $H = \{x - x_0 : x \in E\}$ we obtain that for any $\varepsilon \in (0, \delta)$, Q maps the ε-neighbourhood of x_0 in U onto a neighbourhood of $Q(x_0)$ in $Q(E) \supset Q(U)$; hence $Q(U)$ is (QP) at $Q(x_0)$ by Lemma (7) (ii), thus P_M is pointwise Lipschitz lsc at x_0 by Lemma (7) (i) and Theorem (3).

Next we shall deal with subspaces of $l_1(\Gamma)$ or $l_\infty(\Gamma)$ of a finite codimension. Let M be a closed linear subspace of a finite codimension m in l_1, for example. Then M is the set of all sequences $(x_n) \in l_1$ such that

$$\sum_n a_n^{(i)} x_n = 0 \qquad (i = 1,..., m),$$

where $a_n^{(i)}$ are suitable constants. Let U denote the closed unit ball of l_1 and let $Q: l_1 \to l_1/M$ be the quotient map. Then the set $Q(U)$ can be identified with the set $W \subset R^m$, $W = \{\sum_n x_n b_n : \sum_n |x_n| \leqslant 1\}$, where $b_n = (a_n^{(1)}, a_n^{(2)},..., a_n^{(m)})$ for any n. Using induction on m, it can be proved that W coincides with the convex hull of vectors $\pm b_n$, whence the set $Q(U)$ is easy to represent. In the following Theorem the set $Q(U)$ is used to characterize the lower semicontinuity of P_M in such a case.

(11) THEOREM. *Let $\Gamma \neq \emptyset$ be an arbitrary set and suppose that one of the following cases occurs:*

(i) *$X = l_1(\Gamma)$ and M is a proximinal linear subspace of X of finite codimension;*

(ii) *$X = l_\infty(\Gamma)$ and M is a w^*-closed linear subspace of X of finite codimension.*

Let $Q: X \to X/M$ be the quotient map and let U be the closed unit ball of X. Then P_M is lsc if and only if $Q(U)$ is a polyhedron (i.e., the set of extreme points of $Q(U)$ is finite). Further, if P_M is lsc at some $x \in X$ then P_M is pointwise Lipschitz lsc at x.

We start the proof with a lemma that will be useful later as well.

(12) LEMMA (cf. [9, Lemma 1]). *Let X, M, Q, and U be as in* (11). *Then any extreme point of the set $Q(U)$ is the image of an extreme point of U.*

Proof. In the case (11)(ii) the statement follows immediately from the assertion (3) of [9, Lemma 1]. However, in the case (11)(i) the subspace M is not admissibly compact in the sense of [9] in general (M is not w^* – closed in general); nevertheless, the arguments needed for the proof still work since the space $X = l_1(\Gamma)$ has the so called Krein–Milman property (see, e.g., [11]), i.e., any bounded closed convex nonempty subset of X has at least one extreme point.

Indeed, let z be an extreme point of $Q(U)$, v be an extreme point of the set $V = Q^{-1}(z) \cap U$, and suppose that $v = 1/2(v_1 + v_2)$ for some $v_1, v_2 \in U$. Then $z = 1/2(Q(v_1) + Q(v_2))$. Since z is an extreme point of $Q(U)$ we obtain $v_1, v_2 \in V$ which implies $v_1 = v_2 = v$ as v is an extreme point of V. Hence v is an extreme point of U.

Proof of Theorem (11). The set $K = Q(U)$ is a finite-dimensional compact convex set by Remark (6). If K is a polyhedron then P_M is lsc by Theorem (3), Lemma (7)(i), and Remark (2).

Let P_M be lsc at some $x \in P_M^{-1}(0)$ of norm one. By Carathéodory's theorem, $Q(x)$ is a (finite) convex combination of extreme points of K. Thus, in view of Lemma (12), there is a point $u \in Q^{-1}(Q(x))$ which is a convex combination of extreme points of U. It is easy to see that in both cases (i) and (ii), U is (QP) at any such a point u. Hence K is (QP) at $Q(x)$ and P_M is pointwise Lipschitz lsc at x by Theorem (3) and Lemma (7). Suppose now that K is not a polyhedron. Let y be a cluster point of extreme points of K and let $x \in Q^{-1}(y) \cap U$ be arbitrary. Then K is not (QP) at y, $\|x\| = \|y\| = 1$, and $0 \in P_M(x)$, whence P_M is not lsc at x by the arguments mentioned above. To finish the proof it is enough to use Remark (2).

(13) COROLLARY. *Let M be a w^* – closed linear subspace of $l_1(\Gamma)$ of finite codimension. Then P_M is pointwise Lipschitz lsc.*

Proof. Denote $X = l_1(\Gamma)$. Let Q and U be as in (11). It is enough to show that $K = Q(U)$ is a polyhedron (Theorem (11)). Let y be an extreme point of K. By Lemma (12), $y = Q(\pm e_\gamma)$ for some index γ, where $(e_\gamma)_{\gamma \in \Gamma}$ is the canonical basis of X. Since the only w^*-cluster point of any infinite subset of the canonical basis of X is 0 and Q is continuous from the w^*-topology (M is of finite codimension and w^*-closed in X), we obtain that $\|Q(e_\gamma)\| > 1/2$ for a finite number of indices γ at most. Thus the set of extreme points of K is finite.

In [6] an example of a Čebyšev subspace of l_1 of codimension two with discontinuous metric projection is given. Theorem (3) together with Lemma (7) enable us to construct without tedious computation an example of a Čebyšev subspace M of l_1 of codimension two such that P_M is discontinuous at infinitely many points of $P_M^{-1}(0)$ of norm one. Moreover, we can construct the following

(14) EXAMPLE. Let Γ be a set of the cardinality of continuum. Then there exists a Čebyšev subspace M of $l_1(\Gamma)$ of codimension two such that P_M is discontinuous at any point outside M.

Construction. We can put $\Gamma = [0, \pi)$. Let $(e_\gamma)_{\gamma \in \Gamma}$ be the canonical basis of $X = l_1(\Gamma)$. To define a continuous linear mapping $Q: X \to R^2$ it suffices to set $Q(e_\gamma) = (\cos \gamma, \sin \gamma)$, $\gamma \in \Gamma$. It follows from Remark (6) that $M = Q^{-1}(0)$ is a Čebyšev subspace. It is easy to check that the unit ball of X is (QP) at any point $\pm e_\gamma$ $(\gamma \in \Gamma)$. Obviously, these points are the only points in $P_M^{-1}(0)$ of norm one. Since the circle in R^2 is not (QP) at any point of its boundary, P_M is discontinuous at any point outside M by Theorem (3), Lemma (7) (ii), and Remark (2).

However, in the case $X = l_1$ such an example is impossible since we have the following

(15) PROPOSITION. *Let M be a proximinal linear subspace of l_1 of finite codimension. Then P_M is (pointwise Lipschitz) lsc at any point of a certain set which is open and dense in l_1.*

Proof. We abbreviate $X = l_1$. Let n be the dimension of X/M, Q and U be as in (11). The set $K = Q(U)$ is closed by (6). We denote by ∂K the boundary of K. Let G be the set of the points $y \in \partial K$ such that there exists a hyperplane H supporting K at y such that y is an interior point of $H \cap K$ with respect to H. Obviously G is open in ∂K. We shall prove that G is dense in ∂K. Let $y \in (\partial K) \backslash G$ be arbitrary. We choose a hyperplane H supporting K at y. Since y is a boundary point of at most $(n-1)$-dimensional compact convex set $H \cap K$, y is a convex combination of at most $n-1$ extreme points of K by the theorem of Carathéodory [16, p. 10]. However, the linear span of less than n points meets ∂K in a set nowhere dense in ∂K. By Lemma (12) the set of extreme points of K is countable at most, hence the set $(\partial K) \backslash G$ is of the first category in ∂K and thus it is nowhere dense in ∂K. It is clear that K is (QP) at any point of G and that any point of the set $Q^{-1}(G) \cap U$ belongs to $P_M^{-1}(0)$ because $G \subset \partial K$. Define $E = \{\lambda y: y \in G, \lambda > 0\}$. By Theorem (3), Lemma (7) (i), and Remark (2), P_M is pointwise Lipschitz lsc at any point of the set $Q^{-1}(E)$; this set is open and dense in X since E is open and dense in X/M.

(16) *Remark.* Let M be a Čebyšev linear subspace of a normed linear space X of codimension two. Then P_M is pointwise Lipschitz continuous at any point $x \in P_M^{-1}(0)$ of norm one which is not an extreme point of U (we use the notation as in Theorem (3)). Indeed, if x is such a point, then $Q(x)$

is not an extreme point of a two-dimensional set $Q(U)$ by Remark (6), thus $Q(U)$ is (QP) at $Q(x)$ and we can apply Theorem (3) and Lemma (7) (i). Particulary, if M is a Čebyšev linear subspace of l_1 of codimension two then the set of points $x \in P_M^{-1}(0)$ of norm one such that P_M is discontinuous at x is countable at most. However, there is a Čebyšev linear subspace of l_1 of codimension three for which an analogous assertion fails.

For example, let M be a subspace of l_1 defined by $M = Q^{-1}(0)$, where $Q: l_1 \to R^3$ is the linear continuous mapping defined on the elements of the canonical basis (e_n) of l_1 by

$$Q(e_1) = (1, 0, 0),$$

$$Q(e_2) = (0, 0, 1),$$

and

$$Q(e_n) = \left(\cos\frac{1}{n}, \sin\frac{1}{n}, 0 \right) \qquad \text{for} \quad n = 3, 4, \ldots.$$

Let U denote the closed unit ball of l_1 and $K = Q(U)$. It is not difficult to prove (by the use of support hyperplanes of K) that any point of the boundary of K has exactly one pre-image point in U. Thus, by Remark (6), M is a Čebyšev subspace of l_1. Further, it is elementary to check that U is (QP) at any point which is a finite combination of elements of the basis (e_n) and that K is not (QP) at any point of the segment $\{(t, 0, 1-t): t \in (0, 1]\}$. Hence P_M is discontinuous at any point of the segment $\{te_1 + (1-t)e_2: t \in (0, 1]\}$ by Theorem (3) and Lemma (7) (ii) (observe that this segment lies in $P_M^{-1}(0)$ since its image is contained in the boundary of $Q(U)$).

Finally, we present a specification of Theorem (11) in the case that M is a w^*-closed subspace of l_∞ of codimension two to show that the analogy to Corollary (13) fails in this case.

(17) Example. Let (a_n), (b_n) be absolutely summable sequences of real numbers and M be the subspace of l_∞ consisting of all bounded sequences (x_n) such that $\sum_n a_n x_n = 0$ and $\sum_n b_n x_n = 0$. Then P_M is lsc if and only if the set $\{(a_n, b_n)\} \subset R^2$ does not contain an infinite subset of pairwise independent vectors.

To show this let $Q: l_\infty \to R^2$ be the mapping defined by $Q(x) = (\sum_n a_n x_n, \sum_n b_n x_n)$ for any $x = (x_n) \in l_\infty$. In the case that the codimension of M in l_∞ is one the assertion follows easily; hence suppose that Q is a surjection. If R^2 is given the norm with the unit ball $Q(U)$, where U is the closed unit ball of l_∞, then Q can be identified with the quotient map associated with M. Of course, M is proximinal and $Q(U)$ is closed. Any linear functional f on R^2 can be identified with a pair of real numbers (α, β) so that $f(Q(x)) = \alpha \sum_n a_n x_n + \beta \sum_n b_n x_n = \sum_n (\alpha a_n + \beta b_n) x_n$ for all $x = (x_n) \in l_\infty$. Thus f attains its supremum on $Q(U)$ at $Q(x)$ if and only if $x_n = \operatorname{sign}(\alpha a_n + \beta b_n)$ for any n such that $\alpha a_n + \beta b_n \neq 0$.

Suppose at first that the set $\{(a_n, b_n)\}$ does not contain an infinite subset of pairwise independent vectors. Then there exists a finite set $F \subset R^2$ such that for any $(\alpha, \beta) \in R^2$ there is $(\alpha_0, \beta_0) \in F$ with $\operatorname{sign}(\alpha_0 a_n + \beta_0 b_n) = \operatorname{sign}(\alpha a_n + \beta b_n)$ for any n. Hence for arbitrary $y \in Q(U)$ a functional attaining its supremum on $Q(U)$ at y can be found in the finite set F, thus $Q(U)$ is a polyhedron and P_M is lsc by Theorem (11).

Suppose now that there is an infinite subset of $\{(a_n, b_n)\}$ consisting of pairwise independent vectors. Then there exists an infinite set $F \subset R^2$ such that $\alpha a_n + \beta b_n \neq 0$ for any n and any $(\alpha, \beta) \in F$, and whenever (α_i, β_i) $(i = 1, 2)$ are two different elements of F then $\operatorname{sign}(\alpha_1 a_n + \beta_1 b_n) \neq \operatorname{sign}(\alpha_2 a_n + \beta_2 b_n)$ for at least an index n (consider the angles between the vectors (a_n, b_n) and vectors $(\alpha, \beta) \in R^2$). Suppose that the linear functionals on R^2 which are given by two different elements (α_i, β_i) $(i = 1, 2)$ of F attain their supremum on $Q(U)$ at the same point y. Then for any point $x \in Q^{-1}(y) \cap U$ we have $x_n = \operatorname{sign}(\alpha_i a_n + \beta_i b_n)$ for any n and $i = 1, 2$, which contradicts the definition of F. Since any linear functional on R^2 attains its supremum on $Q(U)$ at an extreme point of $Q(U)$ we obtain that the set of extreme points of $Q(U)$ is infinite so that P_M is not lsc by Theorem (11).

Applications in function spaces

In the last part of the paper we show certain applications of the main theorem and Lemma (7) (i) in function spaces. We present some notes concerning discontinuous metric projections in function spaces at first.

There is a Čebyšev subspace M of codimension two in $C[0, 1]$ with P_M discontinuous [14, Lemma 7.4 on p. 89 and Remark on p. 87], thus P_M fails to be lsc.

Consider now a subspace M of $C[0, 1]$ of the special form $M = \{f \in C[0, 1] : f(s_1) = f(s_2) = 0\}$ for some $s_1, s_2 \in [0, 1]$. Then M is proximinal by Remark (6) since the quotient map Q can be identified with the mapping $Q: C[0, 1] \to R^2$, $Q(f) = (f(s_1), f(s_2))$ for $f \in C[0, 1]$. Moreover, P_M is lsc for any such subspace M (this follows, for instance, from Proposition (19) below because the space $C[0, 1]$ has the property (21)). Example (18) (i) shows that the situation becomes more complicated if we consider the metric projection of a linear *subspace* X of the space of continuous real valued functions on a compact space S onto a proximinal subspace M of X of the form $M = \{f \in X : f(s_1) = f(s_2) = 0\}$ for some s_1, $s_2 \in S$ (it is not known to the author whether such an example as in (18) (i) is possible for $S = [0, 1]$).

Further, it follows, for instance, from the characterization of the so-called (P)-spaces [5] that the space R^3 with the norm $\|(x_1, x_2, x_3)\| = (x_1^2 + x_2^2)^{1/2} + |x_3|$ contains a linear subspace M such that P_M is not lsc. An interesting three-dimensional example such that P_M is not lsc for a one-dimensional subspace M (but P_M has a linear selection) is presented in [10]. In Example (18) (ii) we shall use Corollary (4) to show that the metric projection of the space of polynomials of degree two or less with the supremum norm on $[0, 1]$ onto the subspace generated by x^2 fails to be lsc.

(18) EXAMPLES. (i) There exists a metrizable compact space S, a closed linear subspace X of $C(S)$, and two points $s_1, s_2 \in S$ such that the set $M = \{f \in X: f(s_1) = f(s_2) = 0\}$ is a Čebyšev subspace of X with discontinuous metric projection.

For example, take $X = l_1$ and let $M \subset X$ be a linear Čebyšev subspace of codimension two with discontinuous metric projection (e.g., [6]). Then X can be identified with a closed linear subspace of $C(S)$, where S is the closed unit ball of X^* with the w^*-topology. The topological space S is metrizable since X is separable.

(ii) Let X be the subspace of $C[0,1]$ consisting of all polynomials of degree at most two. Then the metric projection of X onto the one-dimensional space M generated by the polynomial x^2 fails to be lsc.

We identify any polynomial $ax^2 + bx + c$ with the vector (a, b, c). Let $Q: X \to R^2$ be defined $Q((a, b, c)) = (b, c)$. Taking an equivalent norm on R^2 we can assume that Q is the quotient map $X \to X/M$. Let U be the closed unit ball of X. We show that Q is not relatively open on U at $f = (0, 0, 1)$. Let V be a neighbourhood of f in U such that $a > -1$ for any $(a, b, c) \in V$. We claim that $Q(V)$ is not a neighbourhood of $Q(f)$ in $Q(U)$. For any $r \in (0, 1)$ define $f_r \in U$ by $f_r = (-(r+1)^2, 2r(r+1), 1 - r^2)$. Clearly, $Q(f_r)$ converges to $Q(f)$ for r converging to 0 $(r > 0)$. Let $r \in (0, 1)$ and suppose that there is a $g = (a, b, c) \in V$ with $Q(g) = Q(f_r)$. Since $g \in U$, we have $g(1) \leqslant 1$ which gives $a \leqslant -r(r+2)$. Take $x = -a^{-1}r(r+1)$. Then $x \in (0, 1)$ and the assumption $a > -1$ implies $g(x) > 1$, a contradiction. Thus Q is not relatively open on U so that P_M is not lsc by Corollary (4).

We note that in the preceding example the subspace M can be defined, for instance, by $M = \{f \in X: f(0) = 0,\ 4f(1/2) - f(1) = 0\}$. The following Proposition presents a sufficient condition for the lower semicontinuity of P_M in similar cases.

(19) PROPOSITION. *Let S be a set, $S_0 = \{s_j\}_{j=1}^n$ a subset of S consisting of n distinct points, and X be a linear subspace of the linear space of real*

functions on S. Let X be given a norm such that $|f(s)| \leqslant \|f\|$ *for any* $f \in X$ *and* $s \in S_0$.

Let $A = (a_{i,j})$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ *be a real matrix and*

$$M = \left\{ f \in X\colon \sum_{j=1}^{n} a_{i,j} f(s_j) = 0 \text{ for any } 1 \leqslant i \leqslant m \right\}.$$

Suppose that the following condition is satisfied:

> *for any sequence* (ε_j) *with* $|\varepsilon_j| = 1$, $1 \leqslant j \leqslant n$, *there is* $f \in X$ *with* $\|f\| \leqslant 1$ *such that*
>
> $$\sum_{j=1}^{n} a_{i,j} f(s_j) = \sum_{j=1}^{n} a_{i,j} \varepsilon_j \text{ for any } 1 \leqslant i \leqslant m. \tag{20}$$

Then M is proximinal and the metric projection P_M *of X onto M is pointwise Lipschitz lsc.*

Particulary, let X have the property

> *for any sequence* (ε_j) *with* $|\varepsilon_j| = 1$, $1 \leqslant j \leqslant n$, *there is* $f \in X$ *with* $f(s_j) = \varepsilon_j$ *for any* $1 \leqslant j \leqslant n$. (21)

Then M is proximinal and P_M *is pointwise Lipschitz lsc for any matrix A.*

Proof. Let $Q_1 : X \to R^n$ be defined by $Q_1(f) = (f(s_1), ..., f(s_n))$ for $f \in X$, $Q_A : R^n \to R^m$ be defined by $Q_A(x) = Ax^T$ for $x \in R^n$, and let Q be the composed map $Q = Q_A \cdot Q_1$. Denote U the closed unit ball of X and U_0 the closed unit ball of $l_\infty(S_0)$. Of course, the set $K = Q_A(U_0)$ is a polyhedron. It follows from the hypotheses that $Q_1(U) \subset U_0$, hence $Q(U) \subset K$. Let E_0 be the set of extreme points of U_0. Since U_0 is the convex hull of E_0, K is the convex hull of $Q_A(E_0)$. However, the condition (20) means that $Q_A(E_0) \subset Q(U)$. Thus $Q(U) = K$ so that M is proximinal by Remark (6) and P_M is pointwise Lipschitz lsc by Theorem (3), Lemma (7) (i), and Remark (2).

(22) PROPOSITION. *Let* S, S_0, *and* X *be as in Proposition* (19). *Let* $A = (a_{i,j})$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ *be a real matrix such that* $\sum_{j=1}^{n} |a_{i,j}| = 1$ *for any* $1 \leqslant i \leqslant m$, *and suppose for any* $1 \leqslant j \leqslant n$ *there is exactly one index i with* $a_{i,j} \neq 0$. *Let* $M \subset X$ *be defined as in Proposition* (19). *Then the condition* (20) *is equivalent to the following one:*

> *for any sequence* (ε_i) *with* $|\varepsilon_i| = 1$, $1 \leqslant i \leqslant m$, *there is* $f \in X$ *such that* $\sum_{j=1}^{n} a_{i,j} f(s_j) = \varepsilon_i$ *for any* $1 \leqslant i \leqslant m$. (23)

Suppose further that the condition (23) *holds. Let* $f \in X \backslash M$ *be arbitrary. Denote* $b_i = \sum_{j=1}^{n} a_{i,j} f(s_j)$ *for* $1 \leqslant i \leqslant m$, $b = \sup\{|b_i| : 1 \leqslant i \leqslant m\}$, *and*

$\delta = b^{-1} \inf \{b - |b_i|: 1 \leqslant i \leqslant m, |b_i| \neq b\}$ *(put $\delta = 2$ if $|b_i| = b$ for all i). Then for any $g \in X$ and any $m_0 \in P_M(f)$ there is a $m \in P_M(g)$ with $\|m - m_0\| \leqslant 12\delta^{-1}\|f - g\|$ (of course, $\delta > 0$ from the definition).*

Proof. The implication (20) $\Rightarrow$ (23) follows immediately from the assumptions on the matrix A. Let Q_1, Q_A, Q, U, and U_0 be as in the proof of Proposition (19) and let $K = \{y \in R^m: |y_i| \leqslant 1$ for any $1 \leqslant i \leqslant m\}$. We have $Q(U) \subset K$ since $Q_1(U) \subset U_0$ and $Q_A(U_0) \subset K$. The condition (23) means that for any extreme point ε of K there is a $f \in U$ with $Q(f) = \varepsilon$. Thus $Q(U) = K$. Particularly, $Q_A(U_0) \subset Q(U)$ so that the condition (20) is fulfilled. The rest of the assertion follows from Remark (2), Lemma (7) (i), and the proof of Theorem (3) since the set K is (QP) at any point of its boundary with $\delta = \inf \{1 - |y_i|: 1 \leqslant i \leqslant m, |y_i| \neq 1\}$ (we put $\inf \varnothing = 2$).

(24) EXAMPLES. Let S be a compact Hausdorff space, $S_0 = \{s_j\}_{j=1}^n$ a subset of S, and X a linear subspace of $C(S)$. If not stated otherwise, the norm given on X is the usual supremum norm $\|f\| = \sup \{|f(s)|: s \in S\}$, $f \in X$.

(i) $X = C(S)$ satisfies condition (21).

(ii) Let $S = [0, 1]$ and suppose $s_1 < s_2 < \cdots < s_n$. Let $k \geqslant 0$ be an integer and let $C^{(k)}$ denote the space of all real valued functions on S with continuous derivatives of order k or less on S. Suppose that X contains any function $f \in C^{(k)}$ such that f is a polynomial of degree $2k + 1$ or less on any subinterval of S disjoint with S_0. Then X satisfies condition (21).

(iii) Suppose X satisfies condition (21) with certain norm $\| \ \|_0$ and let $p: X \to R$ be an arbitrary pseudonorm. Then there exists *a* $\delta > 0$ such that X satisfies condition (21) with the norm $\|f\|_1 = \sup \{\|f\|_0, \delta p(f)\}, f \in X$. This is clear since we can take $\delta \leqslant (\sup \{p(f): f \in F\})^{-1}$, where $F \subset X$ is a finite set.

Thus, for example, let X be as in (24) (ii) and $X \subset C^{(k)}$. We denote $\alpha = \inf \{s_{j+1} - s_j: 1 \leqslant j < n\}$. Then X satisfies condition (21) with the norm

$$\|f\|_1 = \sup \{\|f\|, 1/3\alpha\|f'\|\} \qquad \text{for } k = 1 \ (f \in X),$$

while for $k = 2$ we can take the norm

$$\|f\|_2 = \sup \{\|f\|, 1/4\alpha\|f'\|, 1/12\alpha^2\|f''\|\}, \qquad f \in X,$$

for example.

(iv) Let $S = [0, 1]$, $S_0 = \{0, 1/2, 1\}$, $k > 1$ be an integer and let X be the space of all polynomials of degree k or less. Then it is easy to show that

X satisfies condition (21) if and only if $k \geqslant 3$. Suppose further that $k = 2$. Let $A = (a_{i,j})$ $(i = 1, 2; j = 1, 2, 3)$ be a real matrix of rank two and

$$M = \{f \in X: a_{i,1}f(0) + a_{i,2}f(1/2) + a_{i,3}f(1) = 0 \text{ for } i = 1, 2\}.$$

Suppose that $a_{1,j}^2 + a_{2,j}^2 \neq 0$ for $j = 1, 2, 3$. For a fixed subspace M defined in this way we can suppose $a_{2,1} = 0$ without loss of generality. Then it is easy to show that the condition (20) is satisfied if and only if $a_{2,3} = 0$ and sign $(a_{1,1}) =$ sign $(a_{1,3})$. We recall that in view of Example (18) (ii), P_M fails to be lsc for a certain matrix A.

Finally, we note that Propositions (19) and (22) can be generalized by taking an arbitrary real normed linear space X and a finite subset S_0 of the closed unit ball of X^*.

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REFERENCES

1. D. AMIR AND F. DEUTSCH, Suns, moons and quasi-polyhedra, *J. Approx. Theory* **6** (1972), 176–201.
2. B. O. BJÖRNESTAL, Local Lipschitz continuity of the metric projection operator, *in* "Approximation Theory" (Papers, VIth Semester, Stefan Banach International Mathematics Center, Warsaw, 1975), pp. 43–53, Banach Center Publ. No. 4, PWN, Warsaw, 1979 (**MR** 80k: 41019).
3. R. C. O'BRIEN, On the openness of the barycentre map, *Math. Ann.* **223** (1976), 207–212.
4. B. BROSOWSKI AND R. WEGMANN, On the lower semicontinuity of the set-valued metric projection, *J. Approx. Theory* **8** (1973), 84–100.
5. A. L. BROWN, Best n-dimensional approximation to sets of functions, *Proc. London Math. Soc.* **14** (1964), 577–594.
6. E. W. CHENEY AND D. E. WULBERT, Existence and unicity of best approximation, *Math. Scand.* **24** (1969), 113–140.
7. A. CLAUSING AND S. PAPADOPOULOU, Stable convex sets and extremal operators, *Math. Ann.* **231** (1978), 193–203.
8. M. M. DAY, "Normed Linear Spaces," 3rd ed., Springer-Verlag, Berlin/Heidelberg/New York, 1973.
9. F. DEUTSCH, Cardinality of extreme points of the unit sphere with applications to non-existence theorems in approximation theory, *in* "Approximation Theory" (G. G. Lorentz Ed.), pp. 319–323, Academic Press, New York, 1973.
10. F. DEUTSCH AND P. S. KENDEROV, Continuous selections and approximate selections for set-valued mappings and applications to metric projections, *SIAM J. Math. Anal.* **14** (1983), 185–194.

11. J. DIESTEL, "Geometry of Banach Spaces—Selected Topics, "Lecture Notes in Mathematics, Vol. 485, Springer-Verlag, Berlin/Heidelberg/New York, 1975.
12. R. HOLMES AND B. KRIPKE, Smoothness of approximation, *Michigan Math. J.* **15** (1968), 225–248.
13. H. KRÜGER, A remark on the lower semicontinuity of the set-valued metric projection, *J. Approx. Theory* **28** (1980), 83–86.
14. J. LINDENSTRAUSS, Extension of compact operators, *Mem. Amer. Math. Soc.* **48** (1964).
15. S. PAPADOPOULOU, On the geometry of stable compact convex sets, *Math. Ann.* **229** (1977), 193–200.
16. R. R. PHELPS, "Lectures on Choquet's Theorem," Van Nostrand, Princeton, N. J., 1966.
17. J. REIF, (P)-sets, quasipolyhedra and stability, *Comment. Math. Univ. Carolin.* **20** (1979), 757–763.
18. J. REIF, On a property of (locally) uniformly convex bodies, to appear.